Non-Noetherian Symmetries for Oscillators in Classical Mechanics and in Field Theory

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Abstract

Infinitely many new conservation laws both for free fields as well as for test fields evolving on a given gravitational background are presented. The conserved currents are constructed using the field theoretical counterpart of a recently discovered non-Noetherian symmetry which gives rise to a new way of solving the classical small oscillations problem. Several examples are discussed.

1 Introduction

Noether's theorem plays a fundamental role in field theory [1]. Besides Noetherian symmetries there are, however, other kinds of symmetry transformations for the field equations which, loosely speaking, do not preserve the variational principle, i.e., they do not satisfy Noether's theorem [2,3,4]. They are non-Noetherian symmetries. Noether theorem gives rise to a conservation law associated to each Noetherian symmetry transformation of a system. On the other hand, non-Noetherian symmetries provide several (and sometimes infinitely many) conservation laws associated to one transformation [3,4,5,6]. In some instances one non-Noetherian symmetry transformation provides enough information to solve completely an n degrees of freedom problem [4]. In order to be more precise let us turn our attention to the small oscillations problem in classical mechanics. The Lagrangian is

$$L = \frac{1}{2}T_{ij}\dot{q}^i\dot{q}^j - \frac{1}{2}V_{ij}q^iq^j \quad i,j = 1,2,....,n \tag{1}$$

with

$$T_{ij} = T_{ji} \ , \qquad V_{ij} = V_{ji} \tag{2}$$

and

$$det\,\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j} = det\,T_{ij} \neq 0 \tag{3}$$

Consider the transformation

$$q'^i = q^i + \delta q^i, \qquad t' = t \tag{4}$$

with

$$\delta q^i = \epsilon (T^{-1}V)^i{}_j q^j \tag{5}$$

It is straightforward to prove that (4)-(5) is a non–Noetherian symmetry transformation for Lagrangian (1) as it maps the space of solutions of its equations of motion into itself (for details, see [4]). As it is well known, energy is conserved for Lagrangian (1) and therefore

$$H_0 = \frac{1}{2}T_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}V_{ij}q^iq^j \tag{6}$$

is a constant of motion. It may be easily proved [2,3,4] that the deformation δH_0 of H_0 along a symmetry transformation δq^i,

$$\delta H_0 = \frac{\partial H_0}{\partial q^i}\delta q^i + \frac{\partial H_0}{\partial \dot{q}^i}\frac{\bar{d}}{dt}(\delta q^i) \tag{7}$$

is also a constant of motion. Thus, we get for the symmetry transformation given by Eqs. (4)-(5) that

$$H_1 = \frac{1}{2}V_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}(VT^{-1}V)_{ij}q^iq^j \tag{8}$$

is a constant of motion. Deforming H_1, so on and so forth we get that, in general,

$$H_s = \frac{1}{2}((VT^{-1})^{s-1}V)_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}((VT^{-1})^sV)_{ij}q^iq^j \tag{9}$$

is a constant of motion for $s \geq 1$. At most n of these constants of motion are functionally independent due to the Cayley-Hamilton theorem. Note that this restriction dissappears in field theory. Furthermore, it may be proved that all these constants are in involution. In the next sections we will obtain the counterpart of these results for different examples in field theory.

2 Free Scalar Field

Consider the scalar field Lagrangian [7]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\mathrm{m}^2\varphi^2 \tag{10}$$

where $\varphi = \varphi(x^\mu)$ is a real scalar field. The equation of motion is

$$\partial_\mu\partial^\mu\varphi + \mathrm{m}^2\varphi = 0 \tag{11}$$

which written in detail reads

$$\frac{\partial^2\varphi}{\partial t^2} = (\nabla^2 - \mathrm{m}^2)\varphi \tag{12}$$

Consider the transformation

$$\delta_1\varphi = \epsilon(\nabla^2 - \mathrm{m}^2)\varphi \equiv \epsilon \mathrm{D}\varphi \tag{13}$$

It is straightforward to prove that $\delta_1\varphi$ satisfies Eq. (12) , i.e.,

$$\frac{\partial^2}{\partial t^2}\delta_1\varphi = \mathrm{D}\delta_1\varphi \tag{14}$$

Therefore, if φ is a solution of Eq. (12), then, φ' given by

$$\varphi' = \varphi + \delta_1\varphi \tag{15}$$

also solves it. The energy-momentum tensor $\mathrm{T}^{\mu\nu}_{(0)}$

$$\mathrm{T}^{\mu\nu}_{(0)} = \varphi^{,\mu}\varphi^{,\nu} - \eta^{\mu\nu}\mathcal{L} \tag{16}$$

is conserved for the scalar field. It is easy to prove that its first deformation given by

$$\mathrm{T}^{\mu\nu}_{(1)} = \frac{1}{2}\left[\varphi^{,\mu}(\mathrm{D}\varphi)^{,\nu} + \varphi^{,\nu}(\mathrm{D}\varphi)^{,\mu} - \eta^{\mu\nu}(\varphi^{,\alpha}(\mathrm{D}\varphi)_{,\alpha} - \mathrm{m}^2\varphi\mathrm{D}\varphi)\right], \tag{17}$$

is also conserved (the factor $\frac{1}{2}$ has been introduced for convenience). The transformations

$$\delta_n\varphi = \epsilon\,\mathrm{D}^n\varphi \quad n = 1, 2, ... \tag{18}$$

are also symmetry transformations for Eq. (11). Therefore, in general,

$$\mathrm{T}^{\mu\nu}_{(n)} = \frac{1}{2}\left[\varphi^{,\mu}(\mathrm{D}^n\varphi)^{,\nu} + \varphi^{,\nu}(\mathrm{D}^n\varphi)^{,\mu} - \eta^{\mu\nu}(\varphi^{,\alpha}(\mathrm{D}^n\varphi)_{,\alpha} - \mathrm{m}^2\varphi\mathrm{D}^n\varphi)\right], \tag{19}$$

is conserved for any n, as it can be readily checked. To understand the physical meaning of $\mathrm{T}^{\mu\nu}_{(n)}$ it is interesting to consider its expression in terms of the Fourier transform of $\varphi(x)$. The solution $\varphi(x)$ of Eq. (11) may be written in terms of $\varphi(k)$ as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^4k\ \delta(k^2 - \mathrm{m}^2)\theta(k^0)(\mathrm{e}^{ikx}\varphi(k) + \mathrm{e}^{-ikx}\varphi^*(k)) \tag{20}$$

where $kx = k_\mu x^\mu$ and the star denotes complex conjugation, then one gets that the energy is

$$\mathrm{P}^0_{(0)} = \int \mathrm{d}^3\mathrm{x}\ \mathrm{T}^{00}_{(0)} = \int \mathrm{d}^3\mathrm{k}\ \mathrm{k}^0\ \varphi^*(\vec{\mathrm{k}})\varphi(\vec{\mathrm{k}}) \tag{21}$$

where $\varphi(\vec{k}) = (2k^0)^{-1/2}\varphi(k)$, with $k^0 = +\sqrt{\vec{k}^2 + m^2}$ and

$$\mathrm{P}^0_{(n)} = \int \mathrm{d}^3x\ \mathrm{T}^{00}_{(n)} = (-1)^n\int \mathrm{d}^3k\ (k^0)^{2n+1}\ \varphi^*(\vec{k})\varphi(\vec{k}) \tag{22}$$

which is a result very similar to the one obtained for the small oscillations problem [4]. We have, therefore obtained infinitely many conservation laws for the free scalar field. Of course, getting infinitely many conserved quantities for the free scalar field is no surprise since the general solution to the problem has been known for a long time. The purpose of discussing the free scalar field is to get a better understanding of the meaning of the non-Noetherian charges, in the next section we will obtain similar conservation laws for a test scalar field evolving on a given static gravitational background which constitutes a more powerful result.

3 Test Scalar Field on a Static Gravitational Background

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \mathrm{m}^2\varphi^2) \tag{23}$$

where the gravitational field is described by the static metric $g_{\mu\nu}$, with determinant g, which satisfies

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0 \tag{24}$$

and

$$g_{0i} = 0 \tag{25}$$

and g is the determinant of the metric. The equation of motion for the scalar field is

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + \sqrt{-g}\mathrm{m}^2\varphi = 0 \tag{26}$$

or, in full detail

$$\ddot{\varphi} = \frac{-g_{00}}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j\varphi) - \mathrm{m}^2 g_{00}\varphi \tag{27}$$

where $\dot{\varphi} = \partial\varphi/\partial x^0$. The Lagrangian is time independent

$$\frac{\partial\mathcal{L}}{\partial x^0} = 0 \tag{28}$$

and therefore energy is conserved

$$\partial_\mu \mathrm{T}^\mu_{(0)0} = 0 \tag{29}$$

with

$$\mathrm{T}^\mu_{(0)0} = \frac{\sqrt{-g}}{2}\left(2g^{\mu\lambda}\partial_\lambda\varphi\dot{\varphi} - \delta^\mu_0(g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - \mathrm{m}^2\varphi^2)\right) \tag{30}$$

Again we may prove that

$$\delta_1\varphi = \epsilon\,\mathcal{D}\varphi \tag{31}$$

is a symmetry transformation for Eq. (27) with

$$\mathcal{D} \equiv \frac{-g_{00}}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ij}\partial_j) - \mathrm{m}^2 g_{00} \tag{32}$$

Therefore, we find that

$$\mathrm{T}^\mu_{(n)0} = \frac{\sqrt{-g}}{2}\left(g^{\mu\lambda}\dot{\varphi}\partial_\lambda\mathcal{D}^n\varphi + g^{\mu\lambda}\partial_\lambda\varphi\mathcal{D}^n\dot{\varphi} - \delta^\mu_0(g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\mathcal{D}^n\varphi - \mathrm{m}^2\varphi\mathcal{D}^n\varphi)\right) \tag{33}$$

is conserved for any n as it can be readily checked. We have thus found infinitely many independent new conservation laws for a scalar field evolving on a static gravitational background. Note that the general solution for Eq. (27) on a Schwarzschild background metric is not known at present.

Consider the Schwarzschild metric [8]

$$g_{\mu\nu} = \mathrm{diag}(1 - \frac{2M}{r}, \frac{-1}{1-\frac{2M}{r}}, -r^2, -\mathrm{r}^2\sin^2\theta) \tag{34}$$

in units such that G=c=1. In this case,

$$\mathcal{D} = \frac{e^{\lambda}}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}(r^2 \sin\theta e^{\lambda}\frac{\partial}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} - m^2 r^2 \sin\theta\right) \tag{35}$$

with $e^{\lambda} = 1 - 2M/r$. We get that

$$T^0_{(0)0} = \frac{r^2 \sin\theta}{2}(e^{-\lambda}(\dot{\varphi})^2 + e^{\lambda}(\varphi_{,r})^2 + \frac{1}{r^2}(\varphi_{,\theta})^2 + \frac{1}{r^2 \sin^2\theta}(\varphi_{,\phi})^2 + m^2\varphi^2) \tag{36}$$

and

$$\begin{aligned} T^0_{(n)0} &= \frac{r^2 \sin\theta}{2}\Big(e^{-\lambda}\dot{\varphi}\mathcal{D}^n\dot{\varphi} + e^{\lambda}\varphi_{,r}(\mathcal{D}^n\varphi)_{,r} + \frac{1}{r^2}\varphi_{,\theta}(\mathcal{D}^n\varphi)_{,\theta} + \\ &\quad + \frac{1}{r^2 \sin^2\theta}\varphi_{,\phi}(\mathcal{D}^n\varphi)_{,\phi} + m^2\varphi\mathcal{D}^n\varphi\Big) \end{aligned} \tag{37}$$

are conserved for all n. Thus we have infinitely many new conservation laws for the scalar field evolving on a gravitational background. In regard to the convergence of the integrals which define the conserved charges associated to $T^0_{(n)0}$, it is straightforward to realize that $\mathcal{D}^n\varphi$ behaves no worse than φ in the limit $r \to \infty$, for the massive case, while it vanishes faster than φ for the massless case. In other words, the new conserved charges behave (at worst) in the same fashion as the usual conserved energy does (and much better in the massless case). This fact may be explicitly verified for the particular case of a massless scalar field of angular momentum and frequency equal to 0. The explicit solution to Eq. (27) is [9]

$$\varphi(r) = \ln(1 - \frac{2M}{r}) \tag{38}$$

as it can be readily verified. From Eq. (37) we have that T^0_0 and $T^0_{(1)0}$ behave as r^{-2} and r^{-5}. For $n > 1$, $T^0_{(n)0}$ converges faster than r^{-5} when $r \to \infty$.
As another example, consider the following metric [10]

$$g_{\mu\nu} = \mathrm{diag}(r^{2\alpha}, -\beta, -r^2, -r^2\sin^2\theta) \tag{39}$$

which has been considered as a model for galactic dark matter dynamics. In Eq. (38) α and β are constants with $\alpha = 2(\gamma-1)/\gamma$ and $\beta = (\gamma^2+4\gamma-4)/\gamma$, where $2 > \gamma > 1$. The Klein-Gordon equation for a massless scalar field evolving on this metric is separable and its solutions are known [11]. The radial part of $\varphi(x^{\mu})$ is

$$R(r) = r^{\frac{2-3\gamma}{2\gamma}}(AJ_{\nu}(\omega z) + BN_{\nu}(\omega z)) \tag{40}$$

where $J_{\nu}(x)$ and $N_{\nu}(x)$ are the Bessel and Neumann functions, ω is the frequency and

$$\nu^2 = \frac{(3\gamma-2)^2/4 + l(l+1)(\gamma^2+4\gamma-4)}{(2-\gamma)^2}, \quad z = \frac{\sqrt{(\gamma^2+4\gamma-4)}}{2-\gamma} r^{\frac{2-\gamma}{\gamma}} \tag{41}$$

Since the metric (38) satisfies Eqs. (24) and (25) we have that (36) and (37) are conserved. The interesting fact in this case, is that if one studies the asymptotic behaviour of solution (39), one finds that $T^0_{\ 0}$ and $T^0_{(n)0}$ when $r \to \infty$ behave as

$$T^0_{\ 0} \approx \frac{1}{r^\alpha} \ , \qquad T^0_{(n)0} \approx \frac{1}{r^{2n(1-\alpha)+\alpha}} \tag{42}$$

Since $1 > \alpha > 0$ this implies that $2n(1-\alpha) + \alpha > 1$ for $n \geq 1$. It is straightforward to realize that the conserved charge associated to $T^0_{\ 0}$ diverges, while the ones linked to $T^0_{(n)0}$ do exist, for $n \geq 1$. Of course the metric is not asymptotically flat, so there is no Poincaré invariance (at infinity). Nevertheless the new conservation laws provide relevant information for the problem at hand.

4 Non–Linear Systems: Burgers Equation

The results we have presented above hold, in general, for linear differential systems. Nevertheless, there are some physically relevant non–linear equations to which our findings may be applied. Burgers equation is one such example. It has been known for some time [12,13] that (the non–linear) Burgers equation may be, in fact, related to a linear equation, which is, of course, tractable using our method. Therefore, even though in an indirect way, we will use our methods to deal with physically relevant non–linear evolution equations. These results may prove, in the future, to be applicable to other non linear systems.
Consider the linear equation

$$u_t + u_{xx} = 0, \tag{43}$$

for the field $u(x,t)$. Here, u_t means partial differentiation of the field u with respect to t, and similarly for the other suffixes. Define the new field $v(x,t)$ by the transformation

$$v \equiv \frac{u_x}{u}. \tag{44}$$

It is a straightforward matter to prove that v satisfies Burgers equation

$$v_t + v_{xx} + (v^2)_x = 0. \tag{45}$$

We have already seen a general algorithm to generate symmetry transformations for linear differential equations. We find that δu defined by

$$\delta u = u_{xxxx}, \tag{46}$$

is a symmetry transformation for Eq. (43). A symmetry transformation δv based on (46) can now be found for Burgers equation (45),

$$\delta v = (v^4 + 6v^2 v_x + 4v v_{xx} + 3{v_x}^2 + v_{xxx})_x. \tag{47}$$

Of course, simpler transformations can also be constructed, but they will usually produce vanishing deformations of the conserved quantities already obtained.
We are not aware of the existence of a Lagrangian for Eq. (43) by itself, i.e., without considering it together with its time reversed counterpart, in which case the construction of the Lagrangian is trivial. Under these considerations, all the symmetry transformations presented in this Section are non–Noetherian.

5 Summary and Conclusions

We have presented non–Noetherian symmetry transformations for oscillators in classical mechanics as well as in field theory which give rise to many conservation laws by deformation of a given conserved quantity. For the classical mechanical case, the symmetry transformation produced enough constants of the motion to completely solve the small oscillations problem. In the case of field theory, we have found infinitely many conserved quantities even for fields interacting with a given background gravitational field. In some cases, this procedure can be extended to physically relevant non–linear equations such as Burgers equation. These results may also be helpful to deal with Eckhaus equation [13]. The method presented here could be used as an alternative way to diagonalize matrices using the procedure described in the classical mechanical case [4], and it also affords a different procedure to deal with differential equations such as the kind which give rise to special functions, for instance. Finally, we should mention that the results presented in this note may be generalized to include electromagnetic–like forces linear in the velocities for the classical mechanical oscillators and the corresponding changes can be introduced in the partial differential equations for the field oscillators.

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